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# CLOSED STREAMLINE, LOW MACH NUMBER FLOW OF A COMPRESSIBLE FLUID

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ABSTRACT. The problem of a steady, non-viscous, low-Mach number flow with closed streamlines is solved. A generalized Bernouilli equation is derived.

## I. Introduction

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It is known that in the steady flow of an incompressible, nonviscous flow, the vortex along a streamline is either constant or proportional to the distance y from the axis of revolution, depending on whether it is a plane flow or a flow having symmetry of revolution. If we set $\Omega = \omega y^{-\alpha}$ where $\omega$ is the intensity of the vortex vector, $\alpha = 0$ in the plane case and $\alpha = 1$ in the case of the flow of revolution, then $\Omega$ will only depend on the stream function $\psi$. The function $\Omega(\Psi)$ is obtained in general by considering the upstream conditions at infinity.

However, if we are dealing with a flow having closed streamlines within the interior of a bounded region $(D)$ which is assumed to be simply connected, there is not a single condition which allows one to determine $\Omega(\Psi)$. Prandtl [1], and Batchelor [2] showed how the viscosity, no matter how small it is, enters and imposes the condition $\Omega$ = constant for the steady flow within $(D)$ when the Reynolds number becomes infinite. It is necessary to assume that the viscous affects strive to zero everywhere within $(D)$ except within a viscous layer located along the boundary $(C)$ of $(D)$.

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* Number in the margin indicate the pagination in the original foreign text.

In the case of a perfect, compressible fluid, i.e., in which the viscosity and thermal conductivity are zero, the entropy S and the total enthalpy H remain constant along the streamline , and $\Omega$ is given by the expression:

$$\Omega = \omega y^{-\alpha} = \rho\left(T\frac{dS}{d\Psi} - \frac{dH}{d\Psi}\right)$$

where $\rho$ is the volume mass density, and T the temperature. In general, $\Omega$ is not constant along a streamline . The functions $S(\Psi)$ and $H(\Psi)$, in a flow having closed streamlines , cannot be determined except by taking into account the cumulative viscosity effects and the thermal conductivity, no matter how small they are. It appears that there is only a general result for these two functions, as is the case for $\Omega$ in the incompressible case.

By limiting ourselves to the case of compressible fluid at low Mach numbers, one could be tempted to directly apply the results of Prandtl and Batchelor. One would postulate that if the Mach number is small the volume mass density is constant to a first approximation. This reasoning is only accurate if it is assumed that the relative variations of the temperature in the flow are of order $M^2$ where M is a characteristic Mach number. However, if no restriction is imposed on the temperature, the volume mass density is variable and it is appropriate to analyze the problem in a basic way. This is what we propose in this note.

## II. General Equations

The dimensional variables are characterized by an asterisk. The non-dimensional variables are defined by the relationships:

$$\boxed{\begin{array}{ccc} \vec{u} = \dfrac{\vec{u}^*}{U^*} & \rho = \dfrac{\rho^*}{\rho_r^*} & T = \dfrac{T^*}{T_r^*} \\ p = \dfrac{p^* - p_r^*}{\rho_r^* U^{*2}} & \vec{x} = \dfrac{\vec{x}^*}{L^*} & \end{array}}$$

In these definitions, $U^*$, $\rho_r^*$ and $p_r^*$ are characteristic values of velocity, volume mass density, and pressure, respectively. The characteristic temperature value is chosen as $T_r^* = p_r^* / (R\rho_r^*)$, assuming that the fluid under consideration is a perfect gas having constant R. $L^*$ is a characteristic dimension of the region ($D$). The characteristic Mach number of the flow is

$$M = \frac{U^*}{a_r^*}$$

where

$$a_r^* = \sqrt{\gamma R T_r^*}$$

is the speed of sound at the temperature $T_r^*$.

A classical method of studying flows at low Mach number consists of writing $\vec{u}$, p, $\rho$ , etc. in the form of expansions in powers of $M^2$. Here we will only consider the first approximation defined by the first term of each of these expansions. It is easy to establish equations which this first approximation satisfies. They are given by the following:

continuity

$$\text{div}\,(\rho\vec{u}) = 0 \tag{1}$$

momentum

$$\rho\left(\vec{\omega} \times \vec{u} + \text{grad}\left(\frac{\vec{u}^2}{2}\right)\right) + \text{grad}\,p = \frac{1}{\mathcal{R}e}\,\text{div}\,\underline{\underline{\tau}} \tag{2}$$

where

$$\underline{\underline{\tau}} = \lambda\,\text{div}\,\vec{u}\,\underline{\underline{I}} + \mu\,\text{def}\,\vec{u}$$
$$\vec{\omega} = \text{rot}\,\vec{u}$$

($\underline{\underline{I}}$ = unit tensor
def $\vec{u}$ = rate of deformation tensor)

energy

$$\rho\vec{u}.\text{grad}\,T = \frac{1}{\mathcal{R}e\,\mathcal{P}r}\,\text{div}\,(k\,\text{grad}\,T) \tag{3}$$

equation of state

$$\rho T = 1. \tag{4}$$

Below it is understood that every variable which is written in these equations is in fact the first term of an expansion in powers of $M^2$, and we have omitted the identification of the first term by the index zero in order to simplify the notation.

The non-dimensional viscosity coefficients $\lambda$, $\mu$ and the non-dimensional thermal conductivity coefficients k, which are only functions of temperature, are defined by the relationships:

$$\mu(T) = \frac{\mu^*(T^*)}{\mu^*(T_r^*)}, \qquad \lambda(T) = \frac{\lambda^*(T^*)}{\mu^*(T_r^*)} \qquad k(T) = \frac{k^*(T^*)}{k^*(T_r^*)}.$$

The Reynolds and Prandtl numbers are defined by: /50

$$\mathcal{R}e = \frac{U^* L_*^* \rho_r^*}{\mu^*(T_r^*)}, \qquad \mathcal{P}r = \frac{\mu^*(T_r^*) C_p^*}{k^*(T_r^*)}.$$

respectively.

It is clear that if small temperature variations are assumed, more accurately if they of order $M^2$, we have T = 1 to a first approximation, and therefore $\rho = 1$ according to (4). We are led to the case of an incompressible fluid flow, and the result of Prandtl-Batchelor applies. On the other hand, if no restriction is made regarding the temperature, the mass volume density is variable, but its variations are not due to temperature variations, as Equation (4) shows.

The continuity equation makes it possible to introduce a stream function $\Psi$ having the following properties, either for plane flow or flow of revolution:

$$\begin{cases} \rho u y^\alpha = \dfrac{\partial \Psi}{\partial y} \\ \rho v y^\alpha = -\dfrac{\partial \Psi}{\partial x} \end{cases} \tag{5}$$

where u and v are the components of the velocity $\vec{u}$ along the axes Ox and Oy, respectively. Ox is the axis of revolution in the case of a flow of revolution, and $\alpha = 0$ in the plane case and $\alpha = 1$ in the flow of revolution case.

Let us now study the solution of the system of Equations (1) through (4) when Re goes towards infinity, acknowledging that the flow within the interior (*D*) strives towards perfect fluid flow except for a boundary layer of thickness $Re^{-1/2}$ located along the boundary (*C*) of (*D*).

Let us consider the limiting case where 1/Re = 0, according to (3) and (4); it can be seen that $\rho$ and T remain constant along a streamline, and we therefore have:

$$\begin{cases} \rho = \rho(\Psi) \\ T = T(\Psi). \end{cases} \tag{6}$$

Let $\vec{s}$ be a unit vector tangent to a streamline, directed along $\vec{u}$. Let us set:

$$\vec{u} = q\,\vec{s} \qquad \text{with} \qquad q = |\vec{u}|.$$

Let $\vec{n}$ be the unit vector perpendicular to $\vec{s}$ in the plane xOy. If we consider (6), and the relationship:

$$\operatorname{grad} \Psi = \rho q y^{\alpha} \vec{n}$$

it can easily be derived from (2), for 1/Re = 0, that the quantity

$$F = \frac{q^2}{2} + \frac{p}{\rho} \tag{7}$$

is constant along a streamline:

$$F = F(\Psi)$$

and the vortex $\omega$ is given by the expression:

$$\Omega = \omega y^{-\alpha} = -\frac{p}{\rho}\frac{d\rho}{d\Psi} - \rho\frac{dF}{d\Psi}. \tag{8}$$

## III. Determination of $\rho(\Psi)$

Just as in the incompressible case in which $\omega(\Psi)$ is determined, in order to determine $\rho(\Psi)$ it is necessary to take into account dissipative effects. The energy Equation (3) can be written as follows by taking into account (1) and (4):

$$\operatorname{div}\left\{\vec{u} - \frac{1}{\mathcal{R}e\ \mathcal{P}r}k(T)\ \mathrm{grad}\ T\right\} = 0.$$

For any closed path $(\Gamma)$ within the interior $(D)$ we have:

$$\oint_{\Gamma} y^{\alpha}\left\{\vec{u} - \frac{1}{\mathcal{R}e\ \mathcal{P}r}k(T)\ \mathrm{grad}\ T\right\}.\vec{n}\ d\Gamma = 0$$

where $\vec{n}$ is a unit normal on $(\Gamma)$. Let us select a streamline for $(\Gamma)$. Thus, $\vec{u}.\vec{n} = 0$, and we find:

$$\oint_{\psi = Cte} y^{\alpha}k(T)\ \mathrm{grad}\ T.\vec{n}\ ds = 0. \tag{9}$$

This relationship is independent of Reynolds number and thus should remain valid when Re strives to infinity. It should be noted that the function k(T) is of order unity and is independent of Re. In a limit, for 1/Re = 0, T is only a function of $\Psi$, and the relationship (9) gives:

$$k(T)\frac{dT}{d\Psi}\oint_{\psi = Cte} y^{2\alpha}\rho q ds = 0. \tag{10}$$

Since the line integral is positive, it is seen that we have of necessity

$$T(\Psi) = \frac{1}{\rho(\Psi)} = C^{te}. \tag{11}$$

The simplicity of this result obviously results in simplifications which can be introduced into the energy Equation (3) and the equation of state (4) due to the hypothesis of the small Mach number. In particular this can be done, because the viscous dissipation is ignored.

Equation (8) shows that $\Omega$ is only a function of $\Psi$ for a perfect fluid:

$$\Omega = -\rho \frac{dF}{d\Psi}. \tag{12}$$

On the other hand, within the interior of the boundary layer along $(C)$, $\rho$ and T are variable and depend on the thermal conditions imposed along $(C)$.

## IV. Determination of $\Omega(\Psi)$

The fact that perfect fluid flow is incompressible does not allow one to conclude that we are led to the case studied by Prandtl and Batchelor and that $\Omega$ is constant. In effect, in order to determine $\Omega(\Psi)$, it is necessary to use a relationship which holds no matter what Re is, and in a viscous fluid the volume mass density is no longer constant.

In order to obtain this kind of relationship, let us multiply (2) scalarly by $\vec{s}$, a unit vector along $\vec{u}$. Let us integrate along a streamline.

We obtain:

$$\begin{aligned} &\oint_{\psi = Cte} \rho \vec{s}.\mathrm{grad}\left(\frac{\vec{u}^2}{2}\right) ds \\ &\qquad = \frac{1}{\mathcal{R}e} \oint_{\psi = Cte} \vec{s}.\mathrm{div}\ (\mu \ \mathrm{def}\ \vec{u})\ ds. \end{aligned} \tag{13}$$

In an incompressible fluid, the first term would be zero, and (13) would be independent of Re. This is not the case here, and in order to make Re vanish from (13) we write:

$$\rho\vec{s}.\mathrm{grad}\left(\frac{\vec{u}^2}{2}\right) = \vec{s}.\mathrm{grad}\left(\rho\frac{\vec{u}^2}{2}\right) - \frac{1}{2} q \vec{u}.\mathrm{grad}\ \rho$$

and we use (3) and (4) for transforming $\vec{u}.\,\mathrm{grad}\,\rho$ : /51

$$\vec{u}.\,\mathrm{grad}\;\rho = -\frac{1}{T}\frac{1}{\mathcal{R}e\;\mathcal{P}r}\,\mathrm{div}\,(k\;\mathrm{grad}\;T).$$

The relationship (13) becomes:

$$\oint_{\psi = Cte} \frac{1}{\mathcal{P}r}\frac{q}{2T}\,\mathrm{div}\,(k\,\mathrm{grad}\,T)\,ds = \oint_{\psi = Cte} \vec{s}.\,\mathrm{div}\,(\mu\,\mathrm{def}\,\vec{u})\,ds \tag{14}$$

where the Reynolds number has been eliminated (the functions k(T) and μ(T) are of order one and are independent of Re).

Since the relationship (14) is valid for any Re, let us apply to a non-viscous fluid flow for which ρ and T, and therefore k and μ are constants.

The first term is zero in this case, and we have

$$\oint_{\psi = Cte} \vec{s}.\,\mathrm{rot}\,\vec{\omega}\;ds = 0 \tag{15}$$

because div $\vec{u}$ = 0. By taking into account the fact that Ω is only a function of Ψ, we find:

$$\vec{s}.\,\mathrm{rot}\,\vec{\omega} = \frac{d\Omega}{d\Psi}\,\rho q y^{2\alpha} + 2\alpha\Omega(\Psi)\,\vec{s}.\,\mathrm{grad}\,x$$

and (15) becomes:

$$\frac{d\Omega}{d\Psi}\oint_{\psi = Cte} \rho q y^{2\alpha}\,ds = 0.$$

We therefore find of necessity:

$$\Omega = \omega y^{-\alpha} = C^{te} \tag{16}$$

in a perfect fluid flow. Equation (12) therefore leads to a generalized Bernouilli equation:

$$p + \frac{1}{2}\rho q^2 + \Omega\Psi = C^{te}. \tag{17}$$

## V. Conclusion

A steady, non-viscous plane or axisymmetric fluid flow of a perfect gas at low Mach number having closed streamlines therefore behaves to a first approximation like an incompressible fluid flow. The volume mass density, the temperature and $\Omega$ are uniform in the non-viscous flow.

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## References

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2. Batchelor, G. S. On Steady Laminar Flow with Closed Streamlines at Large Reynolds Number. Fluid Mech. Vol. I, (July 1956), p. 177.

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